TWO-DIMENSIONAL TEMPERATURE FIELD OF A SEMICONDUCTOR THERMOELECTRIC CONVERTER

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Analytic relations are obtained for constructing the two-dimensional temperature field of a semiconductor thermoelectric converter operating in the steady-state regime.

In recent years increasing use has been made of semiconductor thermoelectric converters working as dc generators, refrigerators and heat pumps.

These converters are designed on the basis of the fundamental equations of heat balance at the cold and hot junctions of a thermoelement obtained from the solution of the one-dimensional problem of temperature distribution in a rod of finite length, along which flows an electric current, the sides of the rod being adiabatically insulated, and its end faces maintained at constant temperatures T and T_0 ($T > T_0$) [1].

However, in certain circumstances this is only a poor approximation. Usually, in assembling thermoelectric converters, the gaps between the individual thermoelements are filled with epoxy resin or some other high-strength polymer, whose heat conductivity is of the same order or only one order less than that of the thermoelectric material. In this case to find the temperature field and obtain the heat balance equations it is necessary to consider the two-dimensional problem.

We shall isolate in the converter an element bounded by two planes passing through the center of the thermoelement and the center of the insulating layer. In view of symmetry, it is sufficient to consider the temperature field of such an element. Obviously, the planes bounding the element in question can be assumed adiabatic. In solving the problem of finding the temperature field of such an element, we shall make the following assumptions, which are usual in the analysis of thermoelectric converters: the temperatures at the hot and cold junctions are constant and equal, respectively, to T and T_0; the resistivity ρ, the heat conductivity of the thermoelectric material, λ_1, and that of the insulating material, λ_2, are independent of temperature; the release of Thomson heat in the thermoelement can be neglected; and the thermo-emf, heat conductivity and electrical conductivity for the p- and n-type semiconductors are the same. Then the temperature field of the isolated element is described by the equations

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} + \frac{i^2 \rho}{\lambda_1} = 0, \quad 0 \leqslant x \leqslant l, \quad 0 \leqslant y \leqslant \delta_1; \quad (1)$$

$$\frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} = 0, \quad 0 \leqslant x \leqslant l, \quad \delta_1 \leqslant y \leqslant \delta \quad (2)$$

with boundary conditions

$$x = 0, \quad 0 \leqslant y \leqslant \delta; \quad T_1(x, y) = T_2(x, y) = T_0; \quad (3)$$

$$x = l, \quad 0 \leqslant y \leqslant \delta; \quad T_1(x, y) = T_2(x, y) = T; \quad (4)$$

$$0 \leqslant x \leqslant l, \quad y = 0; \quad \frac{\partial T_1}{\partial y} = 0; \quad (5)$$

$$0 \leqslant x \leqslant l, \quad y = \delta; \quad \frac{\partial T_2}{\partial y} = 0, \quad (6)$$

$$0 \leqslant x \leqslant l, \quad y = \delta_1; \quad T_1(x, y) = T_2(x, y); \; \lambda_1 \frac{\partial T_1}{\partial y} = \lambda_2 \frac{\partial T_2}{\partial y}. \quad (7)$$

To solve Eqs. (1) and (2), we multiply them by $\sin k\pi \frac{x}{l}\,dx$ and integrate with respect to x from 0 to l. We denote:

$$P_1(\mu_k, y) = \int_0^l T_1(x, y) \sin k\pi \frac{x}{l}\,dx, \quad (8)$$

$$P_2(\mu_k, y) = \int_0^l T_2(x, y) \sin k\pi \frac{x}{l}\,dx, \text{ where } \mu_k = \frac{k\pi}{l}. \quad (9)$$

After transformation of Eqs. (1) and (2) with account for (3), (4), (8) and (9), we get two ordinary linear differential equations of second order:

$$\frac{d^2 P_1}{dy^2} - \mu_k^2 P_1 + \mu_k [T_0 - T(-1)^k] + \frac{i^2 \rho}{\mu_k \lambda_1} [1 - (-1)^k] = 0, \quad (10)$$

$$\frac{d^2 P_2}{dy^2} - \mu_k^2 P_2 + \mu_k [T_0 - T(-1)^k] = 0. \quad (11)$$

After transformation, boundary conditions (5)–(7) are written thus:

$$y = 0, \; \frac{dP_1}{dy} = 0; \quad y = \delta_1, \; P_1(\mu_k, y) = P_2(\mu_k, y);$$
$$y = \delta, \; \frac{dP_2}{dy} = 0; \quad \lambda_1 \frac{dP_1}{dy} = \lambda_2 \frac{dP_2}{dy}. \quad (12)$$

Solution of Eqs. (10) and (11) with boundary conditions (12) gives values of $P_1(\mu_k, y)$ and $P_2(\mu_k, y)$,

$$P_1(\mu_k, y) =$$

$$= \frac{i^2\rho[1-(-1)^k]}{\mu_k^3\lambda_1}\left[1-\frac{\lambda_2 \operatorname{ch}\mu_k y \operatorname{th}\mu_k\delta_2}{\lambda_1 \operatorname{sh}\mu_k\delta_1+\lambda_2\operatorname{ch}\mu_k\delta_1\operatorname{th}\mu_k\delta_2}\right]-$$

$$-\frac{T(-1)^k-T_0}{\mu_k}; \tag{13}$$

$$P_2(\mu_k, y) = \frac{i^2\rho[1-(-1)^k]}{\mu_k^3\lambda_1}\,\frac{\lambda_1 \operatorname{ch}\mu_k(\delta-y)\operatorname{th}\mu_k\delta_1}{\lambda_2\operatorname{sh}\mu_k\delta_2+\lambda_1\operatorname{ch}\mu_k\delta_2\operatorname{th}\mu_k\delta_1}-$$

$$-\frac{T(-1)^k-T_0}{\mu_k}. \tag{14}$$

To determine values of the functions $T_1(x,y)$ and $T_2(x,y)$ we use the inverse finite Fourier sine transformation and pass from the transform of these functions P_1 and P_2 to the inverse transform [3],

$$T_1(x, y) =$$

$$= \frac{2}{l}\sum_{k=1}^{\infty}\left\{\frac{i^2\rho l^3[1-(-1)^k]}{(k\pi^3)^3\lambda_1}\left[1-\left(\lambda_2\operatorname{ch}k\pi\frac{y}{l}\operatorname{th}k\pi\frac{\delta_2}{l}\right)\times\right.\right.$$

$$\left.\times\left(\lambda_1\operatorname{sh}k\pi\frac{\delta_1}{l}+\lambda_2\operatorname{ch}k\pi\frac{\delta_1}{l}\operatorname{th}k\pi\frac{\delta_2}{l}\right)^{-1}\right]-$$

$$\left.-\frac{[T(-1)^k-T_0]l}{k\pi}\right\}\sin k\pi\frac{x}{l}; \tag{15}$$

$$T_2(x, y) =$$

$$= \frac{2}{l}\sum_{k=1}^{\infty}\left\{\frac{i^2\rho l^3[1-(-1)^k]}{(k\pi)^3\lambda_1}\left[\left(\lambda_1\operatorname{ch}k\pi\frac{\delta-y}{l}\operatorname{th}k\pi\frac{\delta_1}{l}\right)\times\right.\right.$$

$$\left.\times\left(\lambda_2\operatorname{ch}k\pi\frac{\delta_2}{l}+\lambda_1\operatorname{ch}k\pi\frac{\delta_2}{l}\operatorname{th}k\pi\frac{\delta_1}{l}\right)^{-1}\right]-$$

$$\left.-\frac{[T(-1)^k-T_0]l}{k\pi}\right\}\sin k\pi\frac{x}{l}. \tag{16}$$

As shown in [2], the solutions thus obtained represent the unknown functions T_1 and T_2 at all points within the region examined except at the boundaries $x = 0$, $x = l$. However, since the values of the functions at these boundaries are given, it is possible to transform the solutions obtained so that they satisfy the entire region in question.

For this purpose we use the known sums of series [4],

$$\sum_{k=1}^{\infty}\frac{1}{k}\sin k\pi\frac{x}{l} = \frac{\pi}{2}\left(1-\frac{x}{l}\right),\quad 0<\frac{\pi x}{l}<2\pi; \tag{17}$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sin k\pi\frac{x}{l} = \frac{\pi}{2}\frac{x}{l},\quad -\pi<\frac{\pi x}{l}<\pi. \tag{18}$$

Using these relations we can write the series

$$T_0\left(1-\frac{x}{l}\right)+T\frac{x}{l} =$$

$$= \sum_{k=1}^{\infty}\frac{2}{k\pi}[T_0+(-1)^{k-1}T]\sin k\pi\frac{x}{l}. \tag{19}$$

Subtracting the obtained series for the function $T_0(1 - x/l) + Tx/l$ term by term from (15) and (16), we obtain the solution satisfying Eqs. (1) and (2) and boundary conditions (3)–(7) over the whole of the region examined:

$$T_1(x, y) = T\frac{x}{l}+T_0\left(1-\frac{x}{l}\right)+\frac{2i^2\rho l^2}{\pi^3\lambda_1}\sum_{k=1}^{\infty}\frac{[1-(-1)^k]}{k^3}\times$$

$$\times\left[1-\left(\lambda_2\operatorname{ch}k\pi\frac{y}{l}\operatorname{th}k\pi\frac{\delta_2}{l}\right)\times\right. \tag{20}$$

$$\left.\times\left(\lambda_1\operatorname{sh}k\pi\frac{\delta_1}{l}+\lambda_2\operatorname{ch}k\pi\frac{\delta_1}{l}\operatorname{th}k\pi\frac{\delta_2}{l}\right)^{-1}\right]\sin k\pi\frac{x}{l};$$

$$T_2(x, y) = T\frac{x}{l}+T_0\left(1-\frac{x}{l}\right)+$$

$$+\frac{2i^2\rho l^2}{\pi^3\lambda_1}\sum_{k=1}^{\infty}\frac{[1-(-1)^k]}{k^3}\times$$

$$\times\left[\left(\lambda_1\operatorname{ch}k\pi\frac{\delta-y}{l}\operatorname{th}k\pi\frac{\delta_1}{l}\right)\times\right. \tag{21}$$

$$\left.\times\left(\lambda_2\operatorname{sh}k\pi\frac{\delta_2}{l}+\lambda_1\operatorname{ch}k\pi\frac{\delta_2}{l}\operatorname{th}k\pi\frac{\delta_1}{l}\right)^{-1}\right]\sin k\pi\frac{x}{l}.$$

The relations obtained can be used to calculate the temperature field of a thermoelectric converter, and also to determine the heat flows to the junctions.

In computing the temperature field, it is sufficient in (20), (21) to retain only the first term of the series, since the latter converge rapidly. In this case the error for the interval of values of the parameters usually adopted in thermoelectric converters is not more than 5%.

We shall formulate the heat balance equations at the junctions (per unit thickness).

It should be kept in mind that the width of the element in relation to which the temperature field has been investigated is one fourth the width of the thermoelectric pair.

At the cold junction

$$Q_0 = 2eIT_0-4n\lambda_1\int_0^{\delta_1}\left.\frac{\partial T_1}{\partial x}\right|_{x=0}dy-4n\lambda_2\int_{\delta_1}^{\delta}\left.\frac{\partial T_2}{\partial x}\right|_{x=0}dy, \tag{22}$$

at the hot junction

$$Q = 2eIT-4n\lambda_1\int_0^{\delta_1}\left.\frac{\partial T_1}{\partial x}\right|_{x=l}dy-4n\lambda_1\int_{\delta_1}^{\delta}\left.\frac{\partial T_2}{\partial x}\right|_{x=l}dy. \tag{23}$$

After evaluation of the integrals in Eqs. (22), (23), we get

$$Q_0 = 2eIT_0-4n\left(\frac{\lambda_1}{l}\delta_1+\frac{\lambda_2}{l}\delta_2\right)(T-T_0)-$$

$$-\frac{8i^2l\rho\delta_1 n}{\pi^2}\sum_{k=1}^{\infty}\frac{[1-(-1)^k]}{k^2}, \tag{24}$$

$$Q = 2eIT - 4n\left(\frac{\lambda_1}{l}\delta_1 + \frac{\lambda_2}{l}\delta_2\right)(T - T_0) + \frac{8i^2 l\rho\delta_1 n}{\pi^2}\sum_{k=1}^{\infty}\frac{[1-(-1)^k]}{k^2}. \quad (25)$$

Since the relations obtained were derived for a thermoelectric converter of unit thickness, we can set

$$S_1 = 4\delta_1 n; \quad S_2 = 4\delta_2 n,$$

where S_1 is the cross-sectional area of the thermoelements in cm^2 and S_2 is the cross-sectional area of the interlayers of insulation in cm^2. Then

$$i^2 l\rho\delta_1 n = \frac{1}{4} I^2 \rho \frac{l}{S_1} = \frac{I^2 R}{4}; \qquad 4n\,\delta_1 \frac{\lambda_1}{l} = K_1; \quad 4n\,\delta_2 \frac{\lambda_2}{l} = K_2.$$

Moreover, we can evaluate the sum of the series in (24), (25):

$$\sum_{k=1}^{\infty}\frac{[1-(-1)^k]}{k^2} = 2\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{(2k-1)^2} + \dots\right] = \frac{\pi^2}{4}.$$

With account for these transformations, the heat balance equations at the junctions may be written:

$$Q_0 = 2eIT_0 - \frac{1}{2} I^2 R - (K_1 + K_2)(T - T_0),$$

$$Q = 2eIT + \frac{1}{2} I^2 R - (K_1 + K_2)(T - T_0).$$

Thus, the rigorous solution shows that when the two-dimensional temperature field is examined, the heat flux due to release of Joule heat is divided into two equal fluxes—to the cold and hot junctions.

It should be noted that this situation will hold only when the gap between the semiconductors is completely filled with insulating material. Otherwise, when the thermoelement is not insulated over its entire height, the heat flux to the junctions due to the release of Joule heat will be divided into two unequal parts.

The equations obtained for the heat balance at the junctions of a thermoelectric converter can be used to determine the optimal width of the insulating layer between the p- and n-type semiconductors under different conditions of heat transfer between the junctions and the medium. Increasing the thickness of the layer increases the total area of the converter and reduces the irreversible temperature drops between the junctions and the surrounding media, which, on the one hand, must lead to an increase in the efficiency of the converter. On the other hand, an increase in the width of the insulating layer leads to an increase in the heat flow from the hot junctions to the cold due to the thermal conductivity of the insulating material, which in turn leads to a reduction in efficiency. Obviously, for each design there is an optimal width of the insulating layer corresponding to maximum efficiency.

NOTATION

Q_0, Q–heat fluxes at cold and hot junctions of converter, W; i–current density, A/cm^2; ρ–resistivity, ohm·cm; λ_1, λ_2–thermal conductivity of semiconductor and insulating material, respectively, W/cm·°C; R–total electrical resistance of thermocouple, ohms; K_1, K_2–total thermal conductivity of thermocouple and insulating layer, respectively, W/deg; e–thermo-emf, V/deg; l–height of thermoelement, cm; δ_1, δ_2–half-width of thermoelement and insulating layer, respectively ($\delta = \delta_1 + \delta_2$); I–current, A; n–number of thermocouples in converter.

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